A SIMPLE METHOD OF CHAOS CONTROL

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It is well-known that some dynamical systems depending on the value of systems'parameters exhibit unpredictable,chaotic behaviour[1- 6].Such a situation makes impossible long-range prediction of system's behavior, but paradoxically allow one to control this behavior with tiny perturbations (see,e.g. [7-12] and references therein). The seminal papers [7-8] induced avalanche of research works in the theory of control of chaos in synergetics.Chaos synchronization in dynamical systems is one of such ways of controlling chaos. According to [7-8] synchronization of two systems occurs when the trajectories of one of the systems will converge to the same values as the other and they will remain in step with each other. For the chaotic systems synchronization is performed by the linking of chaotic systems with a common signal or signals (the so-called drivers): suppose that we have a chaotic dynamical system of three or more state variables (it is well-known that for chaotic behaviour in continous dynamical systems thenumber of state variables should be no smaller than three [3-4]). According to [7-8] in the above mentioned way of chaos control one or some of these state variables can be used as an input to drive a subsystem consisting of remaining state variables and which is a replica of part of the original system.In [7-8] it has been shown that if all the Lyapunov exponents (or the largest Lyapunov exponent) or the real parts of these exponents for the subsystem are negative then the subsystem synchronizes to the chaotic evolution of original system.If the largest subsystem Lyapunov exponent is not negative then as it has been proved in [13] synchronism is also possible if a nonreplica system constructed according some rule is used instead of replica system. The interest to the chaos synchronization in part is due to the application of this phenomenen in secure communications, in modeling of brain activity and recognition processes,etc [7-12]. Also it should be mentioned that this method of chaos control may result in the improved performance of chaotic systems (see e.g.[12] and references therein).As it has been shown in [13] from the application viewpoint using of nonreplica systems has some advantages over the replica approach to the chaos synchronization. The above-mentioned chaos synchronization method [7-8] (replica approach) is applied to different chaotic dynamical systems [7-12]. As it is already underlined recently a new approach-nonreplica approach to chaos synchronization is proposed in [13].A detailed analysis of this paper shows that for high dimensional systems the calculation of Lyapunov exponents in general requires to solve high order algebraic equations or to recourse to the help of numerical simulations.

This paper is dedicated to the chaos synchronization in some dynamical systems of N-dimensionality within the nonreplica approach.In this report a simple method to make all the Lyapunov exponents negative is proposed.This is the main feature of my paper.

Suppose that an autonomous dynamical system under study has N state variables:

$$\begin{aligned}\frac{dx_1}{dt} &= f_1(x_1, x_2, \cdots, x_N, a_1, a_2, \cdots, a_N),\\ \frac{dx_2}{dt} &= f_2(x_1, x_2, \cdots, x_N, a_1, a_2, \cdots, a_N),\\ &\vdots \\ \frac{dx_N}{dt} &= f_N(x_1, x_2, \cdots, x_N, a_1, a_2, \cdots, a_N),\end{aligned} \tag{1}$$

where $x_1, x_2, \cdots, x_N$ are state variables,$f_1, f_2, \cdots, f_N$ are sufficiently smooth functions of $x_1, x_2, \cdots, x_N$ and $a_1, a_2, \cdots, a_N$,and $a_1, a_2, \cdots, a_N$ are parameters. Let

$$x_1^{ss}, x_2^{ss}, \cdots, x_N^{ss}, \tag{2}$$

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be the steady state solutions (fixed points) to the original nonlinear dynamical system (1).Also suppose that for some values of parameters the system (1) behaves chaotically. As it is known from [13] while performing chaos synchronization within replica approach one deals with the response system whose dimensionality is less than the dimensionality of the original nonlinear system.But it is trivial that for high dimensional original nonlinear system even in the case of replica approach response system's dimensionality could be high.Also it is well-known that within nonreplica approach response system's dimensionality is equal to the dimensionality of the original nonlinear system. That is why without loss of generality I will investigate the case of nonreplica approach in order to deal with highest possible dimensionality.As it was already mentioned above,the possibility of chaos synchronization essentially depends on the sign of the Lyapunov exponents.To be more precise, these exponents should be negative.
According to [13], within nonreplica approach the response system contains some arbitrary constants added according to some rule.The presence of these arbitrary constants allows one to be more flexible to achieve chaos synchronization.
Without loss of generality, take the state variable x_1 as a driver.Then using approach developed in [13] I construct the following nonreplica response system(with the superscript"nr"):

$$
\begin{aligned}
\frac{dx_1^{nr}}{dt} &= f_1(x_1, x_2^{nr}, \cdots, x_N^{nr}, a_1, a_2, \cdots, a_N) + \alpha_1(x_1^{nr} - x_1) = F_1, \\
\frac{dx_2^{nr}}{dt} &= f_2(x_1, x_2^{nr}, \cdots, x_N^{nr}, a_1, a_2, \cdots, a_N) + \alpha_2(x_1^{nr} - x_1) = F_2, \\
&\vdots \\
\frac{dx_N^{nr}}{dt} &= f_N(x_1, x_2^{nr}, \cdots, x_N^{nr}, a_1, a_2, \cdots, a_N) + \alpha_N(x_1^{nr} - x_1) = F_N,
\end{aligned}
\tag{3}
$$

(Here it is necessary to underline the following point: in order to construct the response system in fact I added to the right-hand side of the initial nonlinear equations linear terms on the difference of original and response system's variable. It is made, as my task is to demonstrate the simplest way of achiving chaos synchronization: according to [13], in principle one can construct the response system by adding to the original nonlinear system arbitrary functions, which vanish when chaos synchronization is achieved.) The eigenvalues of the Jacobian matrix of the nonreplica system

$$
J = \frac{\partial(F_1, \cdots, F_N)}{\partial(x_1^{nr}, \cdots, x_N^{nr})}, \tag{4}
$$

satisfies the following equation:

$$
\lambda^N + p_1\lambda^{N-1} + p_2\lambda^{N-2} + \cdots + p_N = 0, \tag{5}
$$

where $p_1, p_2, \cdots, p_N$ are in general the functions of the arbitrary constants $\alpha_1, \alpha_2, \cdots, \alpha_N$, parameters $a_1, a_2, \cdots, a_N$, and solutions of the original nonlinear system (1) $x_1(t), x_2(t), \cdots, x_N(t)$.It is well-known that in general case with some exceptions it is highly problematic to find the exact analytical solution of the system of nonlinear equations.This fact creates immense difficulties in the treatment of equation (5) analitically.
As it was mentioned above the task of this paper is to make negative all the roots of equation (5) without the need of performing tedious numerical and analitycal calculations. As the analyses show there are some class of dynamical systems, which could be explored from this point of view. In other words, albeit in general the coefficients $p_1, p_2, \cdots, p_N$ are the functions of time, in some cases the equation (5) could be solved easily .
For example, it is trivial, that in the case of constant Jacobians [13] of the initial nonlinear system these coefficients are time- independent (below as an example one of the Rössler models is

investigated), which allows to treat equation (5) quite easily.But there is a wide class of dynamical systems, chaos synchronization in which can be treated with relative ease even in more general case.I mean dynamical systems with bounded solutions.It is widely known that many dynamical systems with dissipative nature have bounded solutions in the sense that solutions of these systems never goes to infinity. It is well-known that the classical Lorenz system is one of well-studied dissipative dynamical systems with bounded solitions (see, e.g.[1, 2, 4-5]).
Below as an example I will investigate this classical Lorenz model in the relatively unexplored case.
But first I present the more general approach developed for the bounded systems.So, suppose that the original nonlinear system has bounded solutions.As it has been shown by E.N.Lorenz in [14], the dissipative systems of the form

$$\frac{dx_i}{dt} = \sum_{j,k=1}^{N} a_{ijk}x_jx_k - \sum_{j=1}^{N} b_{ij}x_j + c_i, \qquad (6)$$

with the constants chosen so that $\sum a_{ijk}x_ix_jx_k$ vanishesidentically and $\sum b_{ij}x_ix_j$ is positive definite, have bounded solutions.Using the boundedness of the solutions and more crucially the presence of arbitrary constants one can try to make the roots of equation (5) negative without conducting explicit calculations of these roots.
Thus, in general I obtain the equation of N-dimensionality for Lyapunov exponents with coefficients depending on arbitrary constants $\alpha_1, \alpha_2, \cdots, \alpha_N$. Due to the flexibility in choosing the form of nonreplica response system, one will be able to obtain eigenvalue equation (5) with coefficients containing only linear terms on these arbitrary constants.(As it will be clear below from the investigation of one of Rössler models, in the case of such linearity in some cases one can first virtually "choose" any desired negative values for Lyapunov exponents and after that calculate "the right" arbtitrary constants to achive the necessary goal even more easily.)
For this purpose I choose such a nonreplica response system which gives rise to the Jacobian containing all the arbitrary constants along one column.It should be noted that if chaos synchronization is investigated within nonreplica approach and the number of driving variables more than unity then it is possible to obtain algebraic equation of N-th order with coefficients containing also nonlinear terms on the arbitrary constants. One should keep in mind, as a rule the more the number of arbitrary constants,the easier to achive our goal of negative Lyapunov exponents. But without loss of generality and for the sake of simplicity a case of coefficients with linear terms on the arbitrary constants will be studied.In the case of nonlinear terms on the arbitrary constants again due to the flexibility warranted by the form of the nonreplica response system it is possible to choose some of these constants so that coefficients before λ's could contain only linear terms on the arbitrary constants.
So I have some N order algebraic equation.Suppose that λ_i $(i = 1, 2, 3, \cdots, N)$ are roots of this equation.It means that the equation (5) for the eigenvalues of the Jacobian matrix of the nonreplica response system could be presented in the following form:

$$\prod_{i=1}^{N}(\lambda - \lambda_i) = 0 \qquad (7),$$

or

$$\lambda^N + s_1\lambda^{N-1} + s_2\lambda^{N-2} + \cdots + s_N = 0 \qquad (8),$$

where $s_1, s_2, \cdots, s_N$ are functions of $\lambda_1, \lambda_2, \ldots, \lambda_N$:

$$s_1 = (-1)^1 \sum_{i=1}^{N} \lambda_i,$$

$$s_2=(-1)^2\sum_{i=1}^{N}\sum_{j,j>i}^{N}\lambda_i\lambda_j \qquad (9),$$

$$s_3=(-1)^3\sum_{i=1}^{N}\sum_{j,j>i}^{N}\sum_{k,k>j}^{N}\lambda_i\lambda_j\lambda_k,$$

$$\vdots$$

$$s_N=(-1)^N\prod_{i=1}^{N}\lambda_i,$$

Now one has a characteristic equation expressed in two ways:1) equation (8);2) equation (5) obtained from the calculation of eigenvalues of Jacobian matrix of the nonreplica response system.Comparing terms with the same order of λ it is possible to express arbitrary constants in the nonreplica response system through the solutions of the characteristic equation (or vice versa):

$$p_1=s_1,p_2=s_2,\cdots,p_N=s_N, \qquad (10)$$

In the equation (10) by replacing the dynamical variables $x_1,x_2,\cdots,x_N$ by some numbers (for the given value of system's parameters) from within the allowable diapason of values of dynamical variables one obtains the time-independent $p_1,p_2,\cdots,p_N$.(It would be quite reasonable to study the behavior of the characteristic equations' coefficients as a function of bounding (limiting) values for the original nonlinear system; but again the free choice of arbitrary constants in the nonreplica approach allow one effectively to achive the goal even without such an investigation).It is well-known that the necessary and sufficient conditions for the roots of eqs.(8) or (5) to have negative real parts) are the Routh-Hurwitz criteria. (Below upon investigating the examples these conditions will be written explicitly.) As it will be seen from the represented below examples one could quite easily" pick up" the appropriate value and sign for the arbitrary constants in the nonreplica approach to make negative the real parts of the Lyapunov exponents.
Thus the feature of my approach to the chaos synchronization is that for some dynamical systems the possibility of chaos synchronization could be judged without calculating Lyapunov exponents explicitly. This feature of approach could be useful from the application point of view in the sense that the feasibility of synchronization could be established with relative easiness.
Now as the first example consider the following nonlinear chaotical dynamical system.The system under consideration is of the form ([15],the fourth model proposed by Rössler in 1977 the so-called model 1977-1V:

$$\frac{dx}{dt}=-y-z,$$

$$\frac{dy}{dt}=x, \qquad (11)$$

$$\frac{dz}{dt}=a(1-x^2)-bz,$$

According to [15] the dynamical system for values of parameters $a=0.275,b=0.2$ (see [16]) exhibits chaotic behaviour. The system (10) has the following fixed point:

$$x=0,z=ab^-1,y=-ab^{-1}, \qquad (12)$$

The system (11) has three dynamical variables, there is only one nonlinear term of a single variable,namely x. I will consider the case, when x variable is the driver.According to [13] the following form of nonreplica system (with the subscript "nr") is adequate:

$$\frac{dx_{nr}}{dt}=-y_{nr}-z_{nr}+\alpha_1(x_{nr}-x),$$

$$\frac{dy_{nr}}{dt} = x + \alpha_2(x_{nr} - x), \quad (13)$$

$$\frac{dz_{nr}}{dt} = a(1 - x^2) - bz_{nr} + \alpha_3(x_{nr} - x),$$

As the calculations show the eigenvalues of the Jacobian matrix of the system (13) satisfies the following equation:

$$\lambda^3 + \lambda^2(b - \alpha_1) + \lambda(\alpha_2$$
$$+\alpha_3 - b\alpha_1) + b\alpha_2 = 0, \quad (14)$$

Suppose that $\lambda_1, \lambda_2, \lambda_3$ are roots of this equation.Then using the aboveproposed method (comparing the equations (5) and (7)) it is very easy to establish the following relationship between the coefficients of eq.(14) and these roots:

$$p_1 = b - \alpha_1 = -(\lambda_1 + \lambda_2 + \lambda_3) = s_1,$$

$$p_2 = \alpha_2 + \alpha_3 - b\alpha_1$$
$$= \lambda_1\lambda_2 + \lambda_1\lambda_3$$
$$+\lambda_2\lambda_3 = s_2,$$
$$p_3 = b\alpha_2$$
$$= -\lambda_1\lambda_2\lambda_3 = s_3, \quad (15)$$

As $\lambda_1, \lambda_2, \lambda_3$ should be negative, I obtain the following inequalities from the relationships (14):

$$p_1 = b - \alpha_1 > 0,$$

$$p_2 = \alpha_2 + \alpha_3 - b\alpha_1 > 0, \quad (16)$$

$$p_3 = b\alpha_2 > 0,$$

But one should keep in mind that these conditions are not sufficient to have negative roots (or roots with negative real parts).According to Routh- Hurwitz criteria, for roots with negative real parts, additional to (16) condition is required: Namely, the inequality

$$p_1p_2 - p_3 = b(\alpha_3 + \alpha_1^2)$$
$$-\alpha_1(b^2 + \alpha_2 + \alpha_3) > 0, \quad (17)$$

also should take place. (In fact, according to [17] the positiveness of $p_1, p_3, p_1p_2 - p_3$ is sufficient, as the positiveness of p_2 follows from the previous inequalities).
As it can be seen from the relationships (16) and (17), it is quite easy to make Lyapunov exponents negative by choosing positive values for α_2 and quite large negative values for α_1. (Here and below on studying the Lorenz model one should keep in mind that in practice the wide dynamic range for state variables is undesirable and this difficulty can be eliminated by a simple transformation of variables, see,e.g. [9].) As the Jacobian in the case of Rössler model is constant, one can even first choose any desired values for the Lyapunov exponents, after that solve the equation (15) to find arbitrary constants.(For non-constant Jacobians it is rather difficult to do, because within my approach the exact values for time-dependent solutions of the original nonlinear system are not necessarily to be known; also one should aware of nonmonotonic behavior of these solutions.) One can see easily that in the case of linear dependence of the coefficients of the characteristic equations (5) or (8) on the arbitrary constants, the task is the simplest one. Presenting the application of the proposed method one should keep in mind the case of this particular Rössler model is the trivial one in the sense that one deals with the constant Jacobian and therefore the coefficients

before λ's are time-independent.

Here I would like to stress the following conclusion which can be derived from the results of the application of the proposed algorithm to the Rössler model investigated in this report. The studied Rössler model contains three state variable and only one nonlinear term of a single variable x.Considering this variable as a driver one obtains in essence linear response system,which contains three arbitrary constants within nonreplica approach to the chaos synchronization and by choosing these arbitrary constants one can make all the Lyapunov exponents negative and perform synchronization.

Using the algorithm proposed in this report it is easy to arrive at the same conclusion in the general case:Namely if one has a nonlinear dynamical system with an N-dimensional phase space, and if all the nonlinear terms are functions of a single variable x, then it is always possible to find an N- dimensional linear response system, with N arbitrary constants, which will synchronize when driven by x, if the N constants are adjusted to make all eigenvalues of the constant Jacobian matrix negative.The linearity of the response system is highly important in the communications applications from the point of view of exact recovery of transmitted signals(see [18] and references cited therein). Speaking about the communications applications of the chaos synchronization one should also mention that by choosing the arbitrary constants one can make all the Lyapunov exponents not only negative, but also larger in magnitude.This fact also is very important from the application viewpoint. Because, the time required for synchronization to take place depends on the value of the largest Lyapunov exponent.

Now as the second example of application of the proposed method consider the nontrivial case of classical Lorenz dynamical system:

$$\frac{dx}{dt} = \sigma(y-x),$$

$$\frac{dy}{dt} = rx - y - xz, \qquad (18)$$

$$\frac{dz}{dt} = xy - bz,$$

It is well-known that the dynamical system (18) for some values of parameters exhibits chaotic behaviour [1-10]. The adopted values of parameters followed by Lorenz and most other investigators are: $\sigma = 10$ and $b = \frac{8}{3}$.As for the values of r for the chaotic behaviour to occur, according to the linear stability analysis, for the given values of other parameters r must be larger than critical Rayleigh number r_{cr}, see, e.g.[1-3]. At $r > r_{cr}$ the fixed points of the Lorenz system

$$x_{ss} = y_{ss} = \pm(b(r-1))^{\frac{1}{2}},$$

$$z_{ss} = r - 1, \qquad (19)$$

become unstable,and there is a strange attractor over which a chaotic motion takes place.

It is well known that for $\sigma = 10$ and $b = \frac{8}{3}$ the critical Rayleigh number is equal to $r_{cr} = 24.74$. In [8], while investigating the chaos synchronization in Lorenz model the value of $r = 60$ was used. As it was mentioned above, the Lorenz model is a classical example of chaotic behavior in low dimensional nonlinear dynamical systems, and is one of well studied nonlinear systems. Although, the chaos synchronization phenomenen in Lorenz system is also investigated in detail, nevertheless there is some gap in the study of this phenomenon. Namely, the possibility of chaos synchronization in the case of z variable as a driver has not been analyzed thoroughly yet.(To my knowledge, there is only one recent paper [19] addressing this issue.In that paper,the chaos sinchronization in the case of z driving is achieved by considering perturbations of the nonlinear system's parameter, to be more specific the perturbation of the parameter r was considered.In this paper I demonstrate that such a synchronization is possible even without parameter perturbations

within non-replica approach.)
As it was shown in [8], in the case of z variable as a driver synchronization of the response subsystem (x,y) with the original Lorenz system does not occur for the values of parameters $\sigma = 10, b = \frac{8}{3}, r = 60$, as one of the sub-Lyapunov exponents is positive. Here I will apply the proposed method of chaos synchronization to this case.
Thus, consider the z variable as a driver.Then according to [13], in the case of failure of replica approach the following nonreplica system (with the subscript "nr") can be used for synchronization purposes.

$$\frac{dx_{nr}}{dt} = -\sigma x_{nr} + \sigma y_{nr}$$
$$+\alpha_1(z_{nr} - z),$$
$$\frac{dy_{nr}}{dt} = r x_{nr} - y_{nr} - x_{nr} z$$
$$+\alpha_2(z_{nr} - z), \qquad (20)$$
$$\frac{dz_{nr}}{dt} = -bz + x_{nr} y_{nr} + \alpha_3(z_{nr} - z),$$

As the calculations show the eigenvalues of the Jacobian matrix of the system (20) satisfy the following equation:

$$\lambda^3 + \lambda^2(\sigma + 1 - \alpha_3)$$
$$-\lambda(y\alpha_1 + x\alpha_2 + (\sigma + 1)\alpha_3)$$
$$+\sigma(r - z) - \sigma)$$
$$-y(\sigma\alpha_2 + \alpha_1) - \sigma\alpha_3$$
$$-x\alpha_2\sigma - (r - z)(x\alpha_1$$
$$-\alpha_3\sigma) = 0, \quad (21)$$

Here $x(t), y(t), z(t)$ are the solutions of the Lorenz system (18).
According to Routh-Hurwitz criteria, the sufficient and necessary conditions to have roots with negative real parts for the equation (21) can be written as:

$$\sigma + 1 - \alpha_3 > 0,$$

$$-y(\sigma\alpha_2 + \alpha_1) - \sigma\alpha_3$$
$$-\alpha_2\sigma - (r - z)(x\alpha_1$$
$$-\alpha_3\sigma) > 0, \quad (22)$$
$$\sigma(\sigma + 1)(1 - (r - z)) + \alpha_3^2(\sigma + 1)$$
$$-(\sigma + 1)^2\alpha_3$$
$$+\sigma y(\alpha_2 - \alpha_1) + x(\alpha_1(r - z) - \alpha_2)$$
$$+\alpha_3(y\alpha_1 + x\alpha_2) > 0$$

To move further I use the fact that solutions of the Lorenz system is bounded.The bounding value depends on the relationships between the system's parameters and the expression for it could be found in different textbooks and papers, see, e.g.[2, 5, 20]. As the solutions of the initial Lorenz model are bounded and one can choose the magnitude of the arbitrary constants arbitrarily large or small and the sign negative or positive, then it can be seen easily from the equation (22), say, by equalizing α_3 to a large negative value to -100),and by choosing α_1 and α_2 approximately equal in magnitude, it is possible to make negative the real parts of the Lyapunov exponents.

For the obtaining of the negative Lyapunov exponents, it would be quite helpful,to take into account the fact that after transition processes in the long time limit for $\sigma >> 1$, $x(t) \approx y(t)$ and $z(t) > 0$.For the given values of the system's parameters it is relatively easy to "predict" the right order of arbitrary constants to obtain Lyapunov exponents with negative real parts. Really, taking into account the above-mentioned equality of $x(t)$ and $y(t)$, also the positiveness of $z(t)$ and writing $z(t) = r - 1 - \epsilon$, where ϵ is not necessarily a small number, one can obtain the following expressions for the coefficents of the characteristic equation: $a_1 = \sigma + 1 - \alpha_3, a_2 = -((\sigma+1)\alpha_3 + y(\alpha_1+\alpha_2)+\epsilon), a_3 = \alpha_3\sigma\epsilon - 2\sigma\alpha_2 y - y(2+\epsilon)$. From this expressions one can easily establish that larger negative values of α_3 and large positive (if $y < 0$) or negative (if $y > 0$) values of α_2 will help to satisfy the conditions of negativity of real parts of the roots of characteristic equation $a_1 > 0, a_2 > 0, a_3 > 0, a_1a_2 - a_3 > 0$. Also are appropriate the larger negative values of α_3 with the small and close magnitudes of constants α_1, α_2 from the point of view of obtaining of Lyapunov exponents with negative real parts. This "right guess" is confirmed by the numerical simulations. Really, for $\sigma = 10, b = \frac{8}{3}, r = 60$, taking $\alpha_3 = -100, \alpha_1 = -1, \alpha_2 = -1$ from the exact solution of the equation for the Lyapunov exponents (the initial Lorenz model was solved by the fourth-order Runge-Kutta model) I found the following values for the Lyapunov exponents: $\lambda_1 = -2.575, \lambda_2 = -11.000, \lambda_3 = -97.425$. So, just using the boundedness of the dynamical systems (eq.(6)) and applying nonreplica approach I was able to perform chaos synchronization in Lorenz model.In difference to the approach developed in [19], I did't use the system's parameters perturbation.

Speaking about the possibility of replacement of the solutions of the original nonlinear system with some constant values for the calculation of sub- Lyapunov exponents, I would like to stress that such an approach for the first time (to my knowledge) was used by the authors of the paper [10]. Namely, as it was proved in [10] numerically, while calculating the sub-Lyapunov exponents for dynamical systems, whose chaotic behavior has arisen out of instability of fixed points (steady-state solutions) one can replace the solutions of the original nonlinear systems with the steady-state solutions. Moreover, for some of these systems (e.g., for Lorenz system, some of Rössler models) sub-Lyapunov exponents of some of the unstable fixed points appears to govern the locking not only to chaotic orbits, but also to the periodic orbits. In other words, the sub-Lyapunov exponents for the fixed points and the periodic orbit also agree with each other. In the light of these results,it would be quite interesting to try to calculate not only sub-Lyapunov exponents, but also total Lyapunov exponents.

Having this in mind, I also calculated (numerically) the total Lyapunov exponents of the equation (18) by replacing the time- dependent solutions of the Lorenz model with the non-trivial steady-state solutions. As the numerical simulations show in general case total Lyapunov exponents for the time-dependent and steady-state solutions are different. For example, using the above-mentioned values of system's parameters $\sigma = 10, b = \frac{8}{3}, r = 60$ and taking instead of time-dependent solutions of the Lorenz model the non- trivial steady-state solution I obtain the following values for the total Lyapunov exponents: $\lambda_1 = -0.251, \lambda_2 = -11.000, \lambda_3 = -99.75$. As it can be seen in general these total Lyapunov exponents for the cases of time- dependent and time-independent solutions are different(although one can see the sharp difference only between one Lyapunov exponents): the two others are in satisfactory agreement; by the way, this tendency was established also for other values of system's parameters, even for cases when one of total Lyapunov exponents becomes positive. It appears that the satisfactory agreement established between two sub-Lyapunov exponents within the replica approach has some memory-retaining influence on the two of the three of the total Lyapunov exponents in the case of non-replica approach to chaos synchronization;at the same time one should be aware of the fact that these two of the total Lyapunov exponents are different in magnitude from those sub-Lyapunov exponents within the replica approach. The fact that the Lyapunov exponents are different for the cases of time-dependent and steady state solutions to the original Lorenz model could be seen from the following

argument even without explicit numerical and analytical calculations: really for the case of steady state solutions the equation (21) gives the following expression:

$$\lambda^3 + \lambda^2(\sigma + 1 - \alpha_3)$$

$$-\lambda((\alpha_1 + \alpha_2)x_{ss} + \sigma + 1)\alpha_3)$$

$$-2x_{ss}(\sigma\alpha_2 + \alpha_1) = 0, \ (23)$$

It can be seen easily from this equation for $\alpha_1 = 0, \alpha_2 = 0$ one of the roots of this equation is equal to zero exactly.Putting $\alpha_1 = 0, \alpha_2 = 0$ also in the equation (21) will not give the same result. Having in mind the above-mentioned satisfactory agreement between the sub-Lyapunov exponents for the fixed points and the periodic orbits, one can say that in general the total Lyapunov exponents of the chaotic orbit and the periodic orbits also will not coincide with each other. Speaking about the different values for the total Lyapunov exponents,one should also to mention that in some special cases by choosing the appropriate arbitrary constants, one can obtain close total Lyapunov exponents for the cases of both time-dependent and time-independent solutions with high degree of accuracy.

Really, taking $\alpha_1 = -10, \alpha_2 = 0, \alpha_3 = 0$ (with the above-mentioned set of system's parameters) I obtain that in the case of steady-state solutions the real parts of the two Lyapunov exponents are equal to $\lambda_1 = \lambda_2 = -0.326$. The third Lyapunov exponent equals $\lambda_3 = -10.348$. The numerical calculation of total Lyapunov exponents for the time-dependent solutions gives rise to the following values: $\lambda_1 = \lambda_2 = 0.320, \lambda_3 = -10.361$. One can see that there is a quite good agreement between two cases.But these cases could be called "coincidental closeness" and could be explained by the choice of arbitrary constants in the nonreplica approach. One more point to underline:judging by the form of nonreplica response system, it is clear that, in fact I have used the linear feedback method of chaos control.Preserving all the arbitrary constants, one actually makes the task of making Lyapunov exponents negative easier. It is quite interesting to study this problem in more particular cases, say nullifying one or more of these arbitrary constants, in other words feedback scheme works only for part of the state variables.By studying these cases, I found that to make all Lyapunov exponents negative is problematic, even in some cases virtually impossible.For example, taking in equation (21) $\alpha_1 = 0, \alpha_2$, it is quite easy to obtain that in this case chaos synchronization is not realizable by nonreplica approach. This result conforms to the inference of the recent paper [21], whose author used the methods of differential geometry.

In conclusion, in this report I have shown that using the boundedness of the dynamical systems and nonreplica approach, one can make negative the real parts of the Lyapunov exponents without lengthy, cubmersome and tedious numerical and analytical calculations. Also it has been shown that the total Lyapunov exponents calculated for the cases of time- dependent and steady-state solutions to the dynamical systems, whose chaotic behavior has arisen out instability of fixed points, in general are different from each other.Although, it has established that due to the presence of many arbitrary constants in the response system of equations within the nonreplica approach it is quite possible that in some cases these two set of Lyapunov exponents will be identical.

As an example chaos synchronization in the classical Lorenz model and one of Rössler models is investigated. Generalization of some features of chaos synchronization for high dimensional systems with some form of nonlinearities is also discussed.

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